

SUFFICIENT CONDITIONS FOR LOW-RANK MATRIX RECOVERY, TRANSLATED FROM SPARSE SIGNAL RECOVERY

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ABSTRACT. The low-rank matrix recovery (LMR) is a rank minimization problem subject to linear equality constraints, and it arises in many fields such as signal and image processing, statistics, computer vision, system identification and control. This class of optimization problems is \mathcal{NP} -hard and a popular approach replaces the rank function with the nuclear norm of the matrix variable. In this paper, we extend the concept of s -goodness for a sensing matrix in sparse signal recovery (proposed by Juditsky and Nemirovski [Math Program, 2011]) to linear transformations in LMR. Then, we give characterizations of s -goodness in the context of LMR. Using the two characteristic s -goodness constants, γ_s and $\hat{\gamma}_s$, of a linear transformation, not only do we derive necessary and sufficient conditions for a linear transformation to be s -good, but also provide sufficient conditions for exact and stable s -rank matrix recovery via the nuclear norm minimization under mild assumptions. Moreover, we give computable upper bounds for one of the s -goodness characteristics which leads to verifiable sufficient conditions for exact low-rank matrix recovery.

1. INTRODUCTION

The *low-rank matrix recovery* (LMR for short) is a *rank minimization problem* (RMP) with linear constraints, or the *affine matrix rank minimization problem* which is defined as follows:

$$(1) \quad \text{minimize } \text{rank}(X), \quad \text{subject to } \mathcal{A}X = b,$$

where $X \in \mathbb{R}^{m \times n}$ is the matrix variable, and $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ is a linear transformation and $b \in \mathbb{R}^p$. Although specific instances can often be solved with specialized algorithms, the LMR is \mathcal{NP} -hard. A popular approach for solving LMR in the systems and control community is to minimize the trace of a positive semidefinite matrix variable instead of the rank (see, e.g., [2, 28]). A generalization of this approach to non-symmetric matrices introduced by Fazel, Hindi and Boyd [17] is the famous convex relaxation of LMR (1), which is called *nuclear norm minimization* (NNM):

$$(2) \quad \min \|X\|_* \quad \text{s.t. } \mathcal{A}X = b,$$

where $\|X\|_*$ is the *nuclear norm* of X , i.e., the sum of its singular values. When $m = n$ and the matrix $X := \text{Diag}(x)$, $x \in \mathbb{R}^n$, is diagonal, the LMR (1) reduces to *sparse signal recovery* (SSR), which is the so-called *cardinality minimization problem* (CMP):

$$(3) \quad \min \|x\|_0 \quad \text{s.t. } \Phi x = b,$$

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where $\|x\|_0$ denotes the number of nonzero entries in the vector x , $\Phi \in \mathbb{R}^{m \times n}$ is a sensing matrix. A well-known heuristic for SSR is the ℓ_1 -norm minimization relaxation (basis pursuit problem):

$$\min \|x\|_1 \quad \text{s.t. } \Phi x = b,$$

where $\|x\|_1$ is the ℓ_1 -norm of x , i.e., the sum of absolute values of its entries.

The LMR problems have many applications and appeared in the literature of a diverse set of fields including signal and image processing, statistics, computer vision, system identification and control. For more details, see the recent survey paper [33]. LMR and NNM have been the focus of some recent research in optimization community, see, e.g., [1, 4, 11, 14, 23, 24, 25, 26, 32, 33, 35, 37]. Although there are many papers dealing with algorithms for NNM such as interior-point methods, fixed point and Bregman iterative methods and proximal point methods, there are very few papers dealing with the conditions that guarantee the success of the low-rank matrix recovery via NNM. For instance, following the program laid out in the work of Candès and Tao in compressed sensing (CS, see, e.g., [12, 13, 15]), Recht, Fazel and Parrilo [33] provided a certain *restricted isometry property* (RIP) condition on the linear transformation which guarantees the minimum nuclear norm solution is the minimum rank solution. Recht, Xu and Hassibi [35, 34] gave another condition which characterizes a particular property of the null-space of the linear transformation.

In the setting of CS, there are other characterizations of the sensing matrix, under which ℓ_1 -norm minimization can be guaranteed to yield an optimal solution to SSR, in addition to RIP and null-space properties, see, e.g., [16, 18, 19, 20]. In particular, Juditsky and Nemirovski [18] established necessary and sufficient conditions for a sensing matrix to be “*s-good*” to allow for exact ℓ_1 -recovery of sparse signals with s nonzero entries when no measurement noise is present. They also demonstrated that these characteristics, although difficult to evaluate, lead to verifiable sufficient conditions for exact SSR and to efficiently computable upper bounds on those s for which a given sensing matrix is *s-good*. Furthermore, they established instructive links between *s-goodness* and RIP in the CS context. One may wonder whether we can generalize the *s-goodness* concept to LMR and still maintain many of the nice properties as done in [18]. Here, we deal with this issue. Our approach is based on the singular value decomposition (SVD) of a matrix and the partition technique generalized from CS. In the next section, following Juditsky and Nemirovski’s terminology, we propose definitions of *s-goodness* and *G-numbers* of a linear transformation in LMR. We provide some basic properties of *G-numbers*. In Section 3, we characterize *s-goodness* of a linear transformation in LMR via *G-numbers*. We establish the exact and stable LMR results in Section 4. In Section 5, we show that these characteristics lead to verifiable sufficient conditions for exact *s-rank* matrix recovery and to computable upper bounds on those s , for which a given linear transformation is *s-good*. In Section 6, we consider the connection between *s-goodness* and RIP for a linear transformation in LMR. As a byproduct, we obtain the new bound on restricted isometry constant $\delta_{2s} < \sqrt{2} - 1$. As we were in the final stages of the preparation of this paper, Oymak, Mohan, Fazel and Hassibi [31] proposed a general technique for translating results from SSR to LMR, where they give the current best bound on the restricted isometry constant $\delta_{2s} < 0.472$. These results were independently obtained. A difference between the results is that we follow Juditsky and Nemirovski’s geometric, optimization based approach.

Let $W \in \mathbb{R}^{m \times n}$, $r := \min\{m, n\}$ and let $W = U \text{Diag}(\sigma(W)) V^T$ be the SVD of W , where $U \in \mathbb{R}^{m \times r}$, $V \in \mathbb{R}^{n \times r}$, and $\text{Diag}(\sigma(W))$ is the diagonal matrix of $\sigma(W) = (\sigma_1(W), \dots, \sigma_r(W))^T$ which is the vector of the singular values of W . Also let $\Xi(W)$ denote the set of pairs of matrices (U, V) in the SVD of W , i.e.,

$$\Xi(W) := \{(U, V) : U \in \mathbb{R}^{m \times r}, V \in \mathbb{R}^{n \times r}, W = U \text{Diag}(\sigma(W)) V^T\}.$$

For $s \in \{0, 1, 2, \dots, r\}$, we say $W \in \mathbb{R}^{m \times n}$ is a *s-rank matrix* to mean that the rank of W is no more than s . For a *s-rank* matrix W , it is convenient to take $W = U_{m \times s} W_s V_{n \times s}^T$ as its SVD where

$U_{m \times s} \in \mathbb{R}^{m \times s}$, $V_{n \times s} \in \mathbb{R}^{n \times s}$ are orthogonal matrices and $W_s = \text{Diag}((\sigma_1(W), \dots, \sigma_s(W))^T)$. For a vector $y \in \mathbb{R}^p$, let $\|\cdot\|_d$ be the *dual norm* of $\|\cdot\|$ specified by $\|y\|_d := \max_v \{\langle v, y \rangle : \|v\| \leq 1\}$. In particular, $\|\cdot\|_\infty$ is the dual norm of $\|\cdot\|_1$ for a vector. Let $\|X\|$ denote the *spectral or the operator norm* of a matrix $X \in \mathbb{R}^{m \times n}$, i.e., the largest singular value of X . In fact, $\|X\|$ is the dual norm of $\|X\|_*$. Let $\|X\|_F := \sqrt{\langle X, X \rangle} = \sqrt{\text{Tr}(X^T X)}$ be the *Frobenius norm* of X , which is equal to the ℓ_2 -norm of the vector of its singular values. We denote by X^T the *transpose* of X . For a linear transformation $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$, we denote by $\mathcal{A}^* : \mathbb{R}^p \rightarrow \mathbb{R}^{m \times n}$ the *adjoint* of \mathcal{A} .

2. PRELIMINARIES

2.1. Definitions. We first go over some concepts related to s -goodness of the linear transformation in LMR (RMP). These are extensions of those given for SSR (CMP) in [18].

Definition 2.1. Let $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ be a linear transformation and $s \in \{0, 1, 2, \dots, r\}$. We say that \mathcal{A} is s -good, if for every s -rank matrix $W \in \mathbb{R}^{m \times n}$, W is the unique optimal solution to the optimization problem

$$(4) \quad \min_{X \in \mathbb{R}^{m \times n}} \{\|X\|_* : AX = \mathcal{A}W\}.$$

We denote by $s_*(\mathcal{A})$ the largest integer s for which \mathcal{A} is s -good. Clearly, $s_*(\mathcal{A}) \in \{0, 1, \dots, r\}$. To characterize s -goodness we introduce two useful s -goodness constants: γ_s and $\hat{\gamma}_s$, we call γ_s and $\hat{\gamma}_s$ G -numbers.

Definition 2.2. Let $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ be a linear transformation, $\beta \in [0, +\infty]$ and $s \in \{0, 1, 2, \dots, r\}$. Then,

(i) G -number $\gamma_s(\mathcal{A}, \beta)$ is the infimum of $\gamma \geq 0$ such that for every matrix $X \in \mathbb{R}^{m \times n}$ with singular value decomposition $X = U_{m \times s} V_{n \times s}^T$ (i.e., s nonzero singular values, all equal to 1), there exists a vector $y \in \mathbb{R}^p$ such that

$$(5) \quad \|y\|_d \leq \beta \text{ and } \mathcal{A}^* y = U \text{Diag}(\sigma(\mathcal{A}^* y)) V^T,$$

where $U = [U_{m \times s} \ U_{m \times (r-s)}]$, $V = [V_{n \times s} \ V_{n \times (r-s)}]$ are orthogonal matrices, and

$$\sigma_i(\mathcal{A}^* y) \begin{cases} = 1, & \text{if } \sigma_i(X) = 1, \\ \in [0, \gamma], & \text{if } \sigma_i(X) = 0, \end{cases} \quad i \in \{1, 2, \dots, r\}.$$

If there does not exist such y for some X as above, we set $\gamma_s(\mathcal{A}, \beta) = +\infty$.

(ii) G -number $\hat{\gamma}_s(\mathcal{A}, \beta)$ is the infimum of $\gamma \geq 0$ such that for every matrix $X \in \mathbb{R}^{m \times n}$ with s nonzero singular values, all equal to 1, there exists a vector $y \in \mathbb{R}^p$ such that

$$(6) \quad \|y\|_d \leq \beta \text{ and } \|\mathcal{A}^* y - X\| \leq \gamma.$$

If there does not exist such y for some X as above, we set $\gamma_s(\mathcal{A}, \beta) = +\infty$ and to be compatible with the special case given by [18], we write $\gamma_s(\mathcal{A})$, $\hat{\gamma}_s(\mathcal{A})$ instead of $\gamma_s(\mathcal{A}, +\infty)$, $\hat{\gamma}_s(\mathcal{A}, +\infty)$, respectively.

From the above definition, we easily see that the set of values that γ takes is closed. Thus, when $\gamma_s(\mathcal{A}, \beta) < +\infty$, for every matrix $X \in \mathbb{R}^{m \times n}$ with s nonzero singular values, all equal to 1, there exists a vector $y \in \mathbb{R}^p$ such that

$$(7) \quad \|y\|_d \leq \beta \text{ and } \sigma_i(\mathcal{A}^* y) \begin{cases} = 1, & \text{if } \sigma_i(X) = 1, \\ \in [0, \gamma_s(\mathcal{A}, \beta)], & \text{if } \sigma_i(X) = 0, \end{cases} \quad i \in \{1, 2, \dots, r\}.$$

Similarly, for every matrix $X \in \mathbb{R}^{m \times n}$ with s nonzero singular values, all equal to 1, there exists a vector $\hat{y} \in \mathbb{R}^p$ such that

$$(8) \quad \|\hat{y}\|_d \leq \beta \text{ and } \|\mathcal{A}^* \hat{y} - X\| \leq \hat{\gamma}_s(\mathcal{A}, \beta).$$

Observing that the set $\{\mathcal{A}^*y : \|y\|_d \leq \beta\}$ is convex, we obtain that if $\gamma_s(\mathcal{A}, \beta) < +\infty$, then for every matrix X with at most s nonzero singular values and $\|X\| \leq 1$ there exist vectors y satisfying (7) and there exist vectors \hat{y} satisfying (8). Moreover, for a given pair \mathcal{A}, s , $\gamma_s(\mathcal{A}, \beta) = \gamma_s(\mathcal{A})$ and $\hat{\gamma}_s(\mathcal{A}, \beta) = \hat{\gamma}_s(\mathcal{A})$, for all β large enough. However, we would not want β to be very large in some situations, see Section 4. Thus, we need to work out an answer to the question “what is large enough” in our context. Below, we give a simple result in this direction as it was done in the vector case, see Proposition 2 in [18] for details.

Proposition 2.3. *Let $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ be a linear transformation and $\beta \in [0, +\infty]$. Assume that for some $\rho > 0$, the image of the unit $\|\cdot\|_*$ -ball in $\mathbb{R}^{m \times n}$ under the mapping $X \mapsto \mathcal{A}X$ contains the ball $B = \{x \in \mathbb{R}^p : \|x\|_1 \leq \rho\}$. Then for every $s \in \{1, 2, \dots, r\}$,*

$$\beta \geq \frac{1}{\rho} \text{ and } \gamma_s(\mathcal{A}) < 1 \Rightarrow \gamma_s(\mathcal{A}, \beta) = \gamma_s(\mathcal{A}).$$

Proof. Fix $s \in \{1, 2, \dots, r\}$. Let $\gamma := \gamma_s(\mathcal{A}) < 1$. Then for every matrix $W \in \mathbb{R}^{m \times n}$ with its SVD $W = U_{m \times s} V_{n \times s}^T$, there exists a vector $y \in \mathbb{R}^p$ such that

$$\|y\|_d \leq \beta \text{ and } \mathcal{A}^*y = U \text{Diag}(\sigma(\mathcal{A}^*y))V^T,$$

where $U = [U_{m \times s} \ U_{m \times (r-s)}]$, $V = [V_{n \times s} \ V_{n \times (r-s)}]$ are orthogonal matrices, and

$$\sigma_i(\mathcal{A}^*y) \begin{cases} = 1, & \text{if } \sigma_i(W) = 1, \\ \in [0, \gamma], & \text{if } \sigma_i(W) = 0, \end{cases} \quad i \in \{1, 2, \dots, r\}.$$

Clearly, $\|\mathcal{A}^*y\| \leq 1$. That is,

$$1 \geq \|\mathcal{A}^*y\| = \max_{X \in \mathbb{R}^{m \times n}} \{\langle X, \mathcal{A}^*y \rangle : \|X\|_* \leq 1\} = \max_{X \in \mathbb{R}^{m \times n}} \{\langle u, y \rangle : u = \mathcal{A}X, \|X\|_* \leq 1\}.$$

From the inclusion assumption, we obtain that

$$\max_{X \in \mathbb{R}^{m \times n}} \{\langle u, y \rangle : u = \mathcal{A}X, \|X\|_* \leq 1\} \geq \max_{u \in \mathbb{R}^p} \{\langle u, y \rangle : \|u\|_1 \leq \rho\} = \rho \|y\|_\infty = \rho \|y\|_d.$$

Combining the above two strings of relations, we derive the desired conclusion. \square

2.2. Convexity and monotonicity of G -numbers. In order to characterize the s -goodness of a linear transformation \mathcal{A} , we study convexity and monotonicity properties of G -numbers. We begin with the result that G -numbers $\gamma_s(\mathcal{A}, \beta)$ and $\hat{\gamma}_s(\mathcal{A}, \beta)$ are convex nonincreasing functions of β .

Proposition 2.4. *For every linear transformation \mathcal{A} and every $s \in \{0, 1, \dots, r\}$, G -numbers $\gamma_s(\mathcal{A}, \beta)$ and $\hat{\gamma}_s(\mathcal{A}, \beta)$ are convex nonincreasing functions of $\beta \in [0, +\infty]$.*

Proof. We only need to demonstrate that the quantity $\gamma_s(\mathcal{A}, \beta)$ is a convex nonincreasing function of $\beta \in [0, +\infty]$. It is evident from the definition that $\gamma_s(\mathcal{A}, \beta)$ is nonincreasing for given \mathcal{A}, s . It remains to show that $\gamma_s(\mathcal{A}, \beta)$ is a convex function of β . In other words, for every pair $\beta_1, \beta_2 \in [0, +\infty]$, we need to verify that

$$\gamma_s(\mathcal{A}, \alpha\beta_1 + (1 - \alpha)\beta_2) \leq \alpha\gamma_s(\mathcal{A}, \beta_1) + (1 - \alpha)\gamma_s(\mathcal{A}, \beta_2), \quad \forall \alpha \in [0, 1].$$

The above inequality holds immediately if one of β_1, β_2 is $+\infty$. Thus, we may assume $\beta_1, \beta_2 \in [0, +\infty)$. In fact, from the argument around (7) and the definition of $\gamma_s(\mathcal{A}, \cdot)$, we know that for every matrix $X = U \text{Diag}(\sigma(X))V^T$ with s nonzero singular values, all equal to 1, there exist vectors $y_1, y_2 \in \mathbb{R}^p$ such that for $k \in \{1, 2\}$,

$$(9) \quad \|y_k\|_d \leq \beta_k \text{ and } \sigma_i(\mathcal{A}^*y_k) \begin{cases} = 1, & \text{if } \sigma_i(X) = 1, \\ \in [0, \gamma_s(\mathcal{A}, \beta_k)], & \text{if } \sigma_i(X) = 0, \end{cases} \quad i \in \{1, 2, \dots, r\}.$$

It is immediate from (9) that $\|\alpha y_1 + (1 - \alpha)y_2\|_d \leq \alpha\beta_1 + (1 - \alpha)\beta_2$. Moreover, from the above information on the singular values of $\mathcal{A}^*y_1, \mathcal{A}^*y_2$, we may set $\mathcal{A}^*y_k = X + Y_k$, $k \in \{1, 2\}$ such that

$$X^T Y_k = 0, X Y_k^T = 0, \text{rank}(Y_k) \leq r - s, \text{ and } \|Y_k\| \leq \gamma_s(\mathcal{A}, \beta_k).$$

This implies for every $\alpha \in [0, 1]$,

$$X^T [\alpha Y_1 + (1 - \alpha)Y_2] = 0, X [\alpha Y_1 + (1 - \alpha)Y_2]^T = 0,$$

and hence $\text{rank}[\alpha Y_1 + (1 - \alpha)Y_2] \leq r - s$, X and $[\alpha Y_1 + (1 - \alpha)Y_2]$ share the same orthogonal row and column spaces. Thus, noting that $\mathcal{A}^*[\alpha y_1 + (1 - \alpha)y_2] = X + \alpha Y_1 + (1 - \alpha)Y_2$, we obtain that $\|\alpha y_1 + (1 - \alpha)y_2\|_d \leq \alpha\beta_1 + (1 - \alpha)\beta_2$ and

$$\sigma_i(\mathcal{A}^*(\alpha y_1 + (1 - \alpha)y_2)) = \begin{cases} 1, & \text{if } \sigma_i(X) = 1, \\ \sigma_i(\alpha Y_1 + (1 - \alpha)Y_2), & \text{if } \sigma_i(X) = 0, \end{cases}$$

for every $\alpha \in [0, 1]$. Combining this with the fact

$$\|\alpha Y_1 + (1 - \alpha)Y_2\| \leq \alpha\|Y_1\| + (1 - \alpha)\|Y_2\| \leq \alpha\gamma_s(\mathcal{A}, \beta_1) + (1 - \alpha)\gamma_s(\mathcal{A}, \beta_2),$$

we obtain the desired conclusion. \square

The following observation that G -numbers $\gamma_s(\mathcal{A}, \beta), \hat{\gamma}_s(\mathcal{A}, \beta)$ are nondecreasing in s is immediate.

Proposition 2.5. *For every $s' \leq s$, we have $\gamma_{s'}(\mathcal{A}, \beta) \leq \gamma_s(\mathcal{A}, \beta)$, $\hat{\gamma}_{s'}(\mathcal{A}, \beta) \leq \hat{\gamma}_s(\mathcal{A}, \beta)$.*

We further investigate the relationship between the G -numbers $\gamma_s(\mathcal{A}, \beta)$ and $\hat{\gamma}_s(\mathcal{A}, \beta)$. The following result generalizes the second part of Theorem 1 of [18] (and its proof).

Proposition 2.6. *Let $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ be a linear transformation, $\beta \in [0, +\infty]$ and $s \in \{0, 1, 2, \dots, r\}$. Then we have*

$$(10) \quad \gamma := \gamma_s(\mathcal{A}, \beta) < 1 \quad \Rightarrow \quad \hat{\gamma}_s\left(\mathcal{A}, \frac{1}{1 + \gamma}\beta\right) = \frac{\gamma}{1 + \gamma} < \frac{1}{2};$$

$$(11) \quad \hat{\gamma} := \hat{\gamma}_s(\mathcal{A}, \beta) < \frac{1}{2} \quad \Rightarrow \quad \gamma_s\left(\mathcal{A}, \frac{1}{1 - \hat{\gamma}}\beta\right) = \frac{\hat{\gamma}}{1 - \hat{\gamma}} < 1.$$

Proof. Let $\gamma := \gamma_s(\mathcal{A}, \beta) < 1$. Then, for every matrix $Z \in \mathbb{R}^{m \times n}$ with s nonzero singular values, all equal to 1, there exists $y \in \mathbb{R}^p$, $\|y\|_* \leq \beta$, such that $\mathcal{A}^*y = Z + W$, where $\|W\| \leq \gamma$ and W and Z share the same orthogonal row and column spaces. For a given pair Z, y as above, take $\tilde{y} := \frac{1}{1 + \gamma}y$. Then we have $\|\tilde{y}\|_* \leq \frac{1}{1 + \gamma}\beta$ and

$$\|\mathcal{A}^*\tilde{y} - Z\| \leq \max\left\{1 - \frac{1}{1 + \gamma}, \frac{\gamma}{1 + \gamma}\right\} = \frac{\gamma}{1 + \gamma},$$

where the first term under the maximum comes from the fact that \mathcal{A}^*y and Z agree on the subspace corresponding to the nonzero singular values of Z . Therefore, we obtain

$$(12) \quad \hat{\gamma}_s\left(\mathcal{A}, \frac{1}{1 + \gamma}\beta\right) \leq \frac{\gamma}{1 + \gamma} < \frac{1}{2}.$$

Now, we assume that $\hat{\gamma} := \hat{\gamma}_s(\mathcal{A}, \beta) < 1/2$. Fix orthogonal matrices $U \in \mathbb{R}^{m \times r}, V \in \mathbb{R}^{n \times r}$. For an s -element subset J of the index set $\{1, 2, \dots, r\}$, we define a set S_J with respect to orthogonal matrices U, V as

$$S_J := \left\{ x \in \mathbb{R}^r : \exists y \in \mathbb{R}^p, \|y\|_d \leq \beta, \mathcal{A}^*y = U \text{Diag}(\sigma(\mathcal{A}^*y))V^T \text{ where } \sigma_i(\mathcal{A}^*y) \begin{cases} = |x_i|, & \text{if } i \in J, \\ \leq \hat{\gamma}, & \text{if } i \in \bar{J}. \end{cases} \right\}.$$

In the above, \bar{J} denotes the complement of J . It is immediately seen that S_J is a closed convex set in \mathbb{R}^r . As in the proof of Theorem 1 in [18], we have

Claim 1. S_J contains the $\|\cdot\|_\infty$ -ball of radius $(1 - \hat{\gamma})$ centered at the origin in \mathbb{R}^r .

Proof. Note that S_J is closed and convex. Moreover, S_J is the direct sum of its projections onto the pair of subspaces

$$L_J := \{x \in \mathbb{R}^r : x_i = 0, i \in \bar{J}\} \text{ and its orthogonal complement } L_J^\perp = \{x \in \mathbb{R}^r : x_i = 0, i \in J\}.$$

Let Q denote the projection of S_J onto L_J . Then, Q is closed and convex (because of the direct sum property above and the fact that S_J is closed and convex). Note that L_J can be naturally identified with \mathbb{R}^s , and our claim is the image $\bar{Q} \subset \mathbb{R}^s$ of Q under this identification contains the $\|\cdot\|_\infty$ -ball B_s of radius $(1 - \hat{\gamma})$ centered at the origin in \mathbb{R}^s . For a contradiction, suppose B_s is not contained in \bar{Q} . Then there exists $v \in B_s \setminus \bar{Q}$. Since \bar{Q} is closed and convex, by a separating hyperplane theorem, there exists a vector $u \in \mathbb{R}^s$, $\|u\|_1 = 1$ such that

$$u^T v > u^T v' \text{ for every } v' \in \bar{Q}.$$

Let $z \in \mathbb{R}^r$ be defined by

$$z_i := \begin{cases} 1, & i \in J, \\ 0, & \text{otherwise.} \end{cases}$$

By definition of $\hat{\gamma} = \hat{\gamma}_s(\mathcal{A}, \beta)$, for s -rank matrix $U \text{Diag}(z) V^T$, there exists $y \in \mathbb{R}^p$ such that $\|y\|_d \leq \beta$ and

$$\mathcal{A}^* y = U \text{Diag}(z) V^T + W,$$

where W and $U \text{Diag}(z) V^T$ have the same row and column spaces, $\|\mathcal{A}^* y - \text{Diag}(z)\| \leq \hat{\gamma}$ and $\|\sigma(\mathcal{A}^* y) - z\|_\infty \leq \hat{\gamma}$. Together with the definitions of S_J and \bar{Q} , this means that \bar{Q} contains a vector \bar{v} with $|\bar{v}_i - \text{sign}(u_i)| \leq \hat{\gamma}$, $\forall i \in \{1, 2, \dots, s\}$. Therefore,

$$u^T \bar{v} \geq \sum_{i=1}^s |u_i| (1 - \hat{\gamma}) = (1 - \hat{\gamma}) \|u\|_1 = 1 - \hat{\gamma}.$$

By $v \in B_s$ and the definition of u , we obtain

$$1 - \hat{\gamma} \geq \|v\|_\infty = \|u\|_1 \|v\|_\infty \geq u^T v > u^T \bar{v} \geq 1 - \hat{\gamma},$$

where the strict inequality follows from the facts that $\bar{v} \in \bar{Q}$ and u separates v from \bar{Q} . The above string of inequalities is a contradiction, and hence the desired claim holds. \diamond

Using the above claim, we conclude that for every $J \subseteq \{1, 2, \dots, r\}$ with cardinality s , there exists an $x \in S_J$ such that $x_i = (1 - \hat{\gamma})$, $\forall i \in J$. From the definition of S_J , we obtain that there exists $y \in \mathbb{R}^p$ with $\|y\|_d \leq (1 - \hat{\gamma})^{-1} \beta$ such that

$$\mathcal{A}^* y = U \text{Diag}(\sigma(\mathcal{A}^* y)) V^T,$$

where $\sigma_i(\mathcal{A}^* y) = (1 - \hat{\gamma})^{-1} x_i = 1$ if $i \in J$, and $\sigma_i(\mathcal{A}^* y)_i \leq (1 - \hat{\gamma})^{-1} \hat{\gamma}$ if $i \in \bar{J}$. Thus, we obtain that

$$(13) \quad \hat{\gamma}_s := \hat{\gamma}_s(\mathcal{A}, \beta) < \frac{1}{2} \Rightarrow \gamma_s \left(\mathcal{A}, \frac{1}{1 - \hat{\gamma}} \beta \right) \leq \frac{\hat{\gamma}}{1 - \hat{\gamma}} < 1.$$

To conclude the proof, we need to prove that the inequalities we established:

$$\hat{\gamma}_s \left(\mathcal{A}, \frac{1}{1 + \hat{\gamma}} \beta \right) \leq \frac{\gamma}{1 + \gamma} \text{ and } \gamma_s \left(\mathcal{A}, \frac{1}{1 - \hat{\gamma}} \beta \right) \leq \frac{\hat{\gamma}}{1 + \hat{\gamma}}$$

are both equations. This is straightforward by an argument similar to the one in the proof of Theorem 1 in [18]. We omit it for the sake of brevity. \square

We end this section by giving an equivalent representation of the G -number $\hat{\gamma}_s(\mathcal{A}, \beta)$. The next result generalizes Theorem 2 of [18] (and its proof). We define a compact convex set first:

$$P_s := \{Z \in \mathbb{R}^{m \times n} : \|Z\|_* \leq s, \|Z\| \leq 1\}.$$

Theorem 2.7. *Let \mathcal{A} be a linear transformation, $\beta \in [0, +\infty]$ and $s \in \{0, 1, \dots, r\}$. Also let P_s be as defined above. Then,*

$$(14) \quad \hat{\gamma}(\mathcal{A}, \beta) = \max_{Z, X} \{ \langle Z, X \rangle - \beta \|\mathcal{A}X\| : Z \in P_s, \|X\|_* \leq 1 \}.$$

Moreover,

$$(15) \quad \hat{\gamma}(\mathcal{A}) = \max_{Z, X} \{ \langle Z, X \rangle : Z \in P_s, \|X\|_* \leq 1, \mathcal{A}X = 0 \}.$$

Proof. Let $B_\beta := \{y \in \mathbb{R}^p : \|y\|_d \leq \beta\}$ and $B := \{X \in \mathbb{R}^{m \times n} : \|X\| \leq 1\}$. By definition, $\hat{\gamma}_s(\mathcal{A}, \beta)$ is the smallest γ such that the closed convex set $C_{\gamma, \beta} := \mathcal{A}^*B_\beta + \gamma B$ contains all matrices with s nonzero singular values, all equal to 1. Equivalently, $C_{\gamma, \beta}$ contains the convex hull of these matrices, namely, P_s . Note that γ satisfies the inclusion $P_s \subseteq C_{\gamma, \beta}$ if and only if for every $X \in \mathbb{R}^{m \times n}$,

$$(16) \quad \begin{aligned} \max_{Z \in P_s} \langle Z, X \rangle &\leq \max_{Y \in C_{\gamma, \beta}} \langle Y, X \rangle = \max_{y \in \mathbb{R}^p, W \in \mathbb{R}^{m \times n}} \{ \langle X, \mathcal{A}^*y \rangle + \gamma \langle X, W \rangle : \|y\|_d \leq \beta, \|W\| \leq 1 \} \\ &= \beta \|\mathcal{A}X\| + \gamma \|X\|_*. \end{aligned}$$

For the above, we adopt the convention that whenever $\beta = +\infty$, $\beta \|\mathcal{A}X\|$ is defined to be $+\infty$ or 0 depending on whether $\|\mathcal{A}X\| > 0$ or $\|\mathcal{A}X\| = 0$. Thus, $P_s \subseteq C_{\gamma, \beta}$ if and only if $\max_{Z \in P_s} \{ \langle Z, X \rangle - \beta \|\mathcal{A}X\| \} \leq \gamma \|X\|_*$. Using the homogeneity of this last relation with respect to X , the above is equivalent to

$$\max_{Z, X} \{ \langle Z, X \rangle - \beta \|\mathcal{A}X\| : Z \in P_s, \|X\|_* \leq 1 \} \leq \gamma.$$

Therefore, the desired conclusion holds. \square

3. S -GOODNESS AND G -NUMBERS

We first give the following characterization result of s -goodness of a linear transformation \mathcal{A} via the G -number $\gamma_s(\mathcal{A})$, which explains the importance of $\gamma_s(\mathcal{A})$ in LMR. In the case of SSR, it reduces to Theorem 1(i) in [18].

Theorem 3.1. *Let $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ be a linear transformation, and s be an integer $s \in \{0, 1, 2, \dots, r\}$. Then \mathcal{A} is s -good if and only if $\gamma_s(\mathcal{A}) < 1$.*

Proof. Suppose \mathcal{A} is s -good. Let $W \in \mathbb{R}^{m \times n}$ be a matrix of rank $s \in \{1, 2, \dots, r\}$. Without loss of generality, let $W = U_{m \times s} W_s V_{n \times s}^T$ be its SVD where $U_{m \times s} \in \mathbb{R}^{m \times s}$, $V_{n \times s} \in \mathbb{R}^{n \times s}$ are orthogonal matrices and $W_s = \text{Diag}((\sigma_1(W), \dots, \sigma_s(W))^T)$. By the definition of s -goodness of \mathcal{A} , W is the unique solution to the optimization problem (4). Using the first order optimality conditions, we obtain that there exists $y \in \mathbb{R}^p$ such that the function $f_y(x) = \|X\|_* - y^T[\mathcal{A}X - \mathcal{A}W]$ attains its minimum value over $X \in \mathbb{R}^{m \times n}$ at $X = W$. So, $0 \in \partial f_y(W)$, or $\mathcal{A}^*y \in \partial \|W\|_*$. Using the fact (see, e.g., [38])

$$\partial \|W\|_* = \{U_{m \times s} V_{n \times s}^T + M : W \text{ and } M \text{ have orthogonal row and column spaces, and } \|M\| \leq 1\},$$

it follows that there exist matrices $U_{m \times (r-s)}, V_{n \times (r-s)}$ such that $\mathcal{A}^*y = U \text{Diag}(\sigma_i(\mathcal{A}^*y)) V^T$ where $U = [U_{m \times s} \ U_{m \times (r-s)}]$, $V = [V_{n \times s} \ V_{n \times (r-s)}]$ are orthogonal matrices and

$$\sigma_i(\mathcal{A}^*y) \begin{cases} = 1, & \text{if } i \in J, \\ \in [0, 1], & \text{if } i \in \bar{J}, \end{cases}$$

where $J := \{i : \sigma_i(W) \neq 0\}$ and $\bar{J} := \{1, 2, \dots, r\} \setminus J$. Therefore, the optimal objective value of the optimization problem

$$(17) \quad \min_{y, \gamma} \left\{ \gamma : \mathcal{A}^*y \in \partial \|W\|_*, \sigma_i(\mathcal{A}^*y) \begin{cases} = 1, & \text{if } i \in J, \\ \in [0, \gamma], & \text{if } i \in \bar{J}, \end{cases} \right\}$$

is at most one. For the given W with its SVD $W = U_{m \times s} W_s V_{n \times s}^T$, let

$$\Pi := \text{conv}\{M \in \mathbb{R}^{m \times n} : \text{the SVD of } M \text{ is } M = [U_{m \times s} \bar{U}_{m \times (r-s)}] \begin{pmatrix} 0_s & 0 \\ 0 & \sigma(M) \end{pmatrix} [V_{n \times s} \bar{V}_{n \times (r-s)}]^T\}.$$

It is easy to see that Π is a subspace and its normal cone (in the sense of variational analysis, see, e.g., [36] for details) is specified by Π^\perp . Thus, the above problem (17) is equivalent to the following convex optimization problem with set constraint

$$(18) \quad \min_{y, M} \{ \|M\| : \mathcal{A}^* y - U_{m \times s} V_{n \times s}^T - M = 0, M \in \Pi \}.$$

We will show that the optimal value is less than 1. For a contradiction, suppose that the optimal value is one. Then, by Theorem 10.1 and Exercise 10.52 in [36], there exist Lagrange multiplier $D \in \mathbb{R}^{m \times n}$ such that the function

$$L(y, M) = \|M\| + \langle D, \mathcal{A}^* y - U_{m \times s} V_{n \times s}^T - M \rangle + \delta_\Pi(M)$$

has unconstrained minimum in y, M equal to 1, where $\delta_\Pi(\cdot)$ is the indicator function of Π . Let y^*, M^* be an optimal solution. Then, by the optimality condition $0 \in \partial L$, we obtain that

$$0 \in \partial_y L(y^*, M^*), \text{ and } 0 \in \partial_M L(y^*, M^*).$$

Direct calculation yields that

$$\mathcal{A}D = 0, \text{ and } 0 \in -D + \partial\|M^*\| + \Pi^\perp.$$

Notice that Corollary 6.4 in [22] implies that for every $C \in \partial\|M^*\|$, $C \in \Pi$ and $\|C\|_* \leq 1$. Then there exist $D_J \in \Pi^\perp$ and $D_{\bar{J}} \in \partial\|M^*\| \subset \Pi$ such that $D = D_J + D_{\bar{J}}$ with $\|D_{\bar{J}}\|_* \leq 1$. Therefore, $\langle D, U_{m \times s} V_{n \times s}^T \rangle = \langle D_J, U_{m \times s} V_{n \times s}^T \rangle$ and $\langle D, M^* \rangle = \langle D_{\bar{J}}, M^* \rangle$. Moreover, $\langle D_{\bar{J}}, M^* \rangle \leq \|M^*\|$ by the definition of the dual norm of $\|\cdot\|$. This together with the facts $\mathcal{A}D = 0$, $D_J \in \Pi^\perp$ and $D_{\bar{J}} \in \partial\|M^*\| \subset \Pi$ yields

$$\begin{aligned} L(y^*, M^*) &= \|M^*\| - \langle D_{\bar{J}}, M^* \rangle + \langle D, \mathcal{A}^* y^* \rangle - \langle D_J, U_{m \times s} V_{n \times s}^T \rangle + \delta_\Pi(M^*) \\ &\geq -\langle D_J, U_{m \times s} V_{n \times s}^T \rangle + \delta_\Pi(M^*). \end{aligned}$$

Thus, the minimum value of $L(y, M)$ is attained, $L(y^*, M^*) = -\langle D_J, U_{m \times s} V_{n \times s}^T \rangle$, when $M^* \in \Pi$, $\langle D_{\bar{J}}, M^* \rangle = \|M^*\|$. We obtain that $\|D_{\bar{J}}\|_* = 1$. By assumption, $1 = L(y^*, M^*) = -\langle D_J, U_{m \times s} V_{n \times s}^T \rangle$. That is, $\sum_{i=1}^s (U_{m \times s}^T D V_{n \times s})_{ii} = -1$. Without loss of generality, let SVD of the optimal M^* be $M^* = \tilde{U} \begin{pmatrix} 0_s & 0 \\ 0 & \sigma(M^*) \end{pmatrix} \tilde{V}^T$, where $\tilde{U} := [U_{m \times s} \tilde{U}_{m \times (r-s)}]$ and $\tilde{V} := [V_{n \times s} \tilde{V}_{n \times (r-s)}]$. From the above arguments, we obtain that

- i) $\mathcal{A}D = 0$,
- ii) $\sum_{i=1}^s (U_{m \times s}^T D V_{n \times s})_{ii} = \sum_{i \in J} (\tilde{U}^T D \tilde{V})_{ii} = -1$,
- iii) $\sum_{i \in \bar{J}} (\tilde{U}^T D \tilde{V})_{ii} = 1$.

Clearly, for every $t \in \mathbb{R}$, the matrices $X_t := W + tD$ are feasible in (4). Note that

$$W = U_{m \times s} W_s V_{n \times s}^T = [U_{m \times s} \tilde{U}_{m \times (r-s)}] \begin{pmatrix} W_s & 0 \\ 0 & 0 \end{pmatrix} [V_{n \times s} \tilde{V}_{n \times (r-s)}]^T.$$

Then, $\|W\|_* = \|\tilde{U}^T W \tilde{V}\|_* = \text{Tr}(\tilde{U}^T W \tilde{V})$. From the above equations, we obtain that $\|X_t\|_* = \|W\|_*$ for all small enough $t > 0$ (since $\sigma_i(W) > 0$, $i \in \{1, 2, \dots, s\}$). Noting that W is the unique optimal solution to (4), we have $X_t = W$, which means that $(\tilde{U}^T D \tilde{V})_{ii} = 0$ for $i \in J$. This is a contradiction, and hence the desired conclusion holds.

We next prove that \mathcal{A} is s -good if $\gamma_s(\mathcal{A}) < 1$. That is, we let W be an s -rank matrix and we show that W is the unique optimal solution to (4). Without loss of generality, let W be a matrix of rank $s' \neq 0$ and $U_{m \times s'} W_{s'} V_{n \times s'}^T$ be its SVD, where $U_{m \times s'} \in \mathbb{R}^{m \times s'}$, $V_{n \times s'} \in \mathbb{R}^{n \times s'}$ are orthogonal matrices and $W_{s'} = \text{Diag}((\sigma_1(W), \dots, \sigma_{s'}(W))^T)$. It follows from Proposition

2.5 that $\gamma_{s'}(\mathcal{A}) \leq \gamma_s < 1$. By the definition of $\gamma_s(\mathcal{A})$, there exists $y \in \mathbb{R}^p$ such that $\mathcal{A}^*y = U \text{Diag}(\sigma(\mathcal{A}^*y))V^T$, where $U = [U_{m \times s'} \ U_{m \times (r-s')}]$, $V = [V_{n \times s'} \ V_{n \times (r-s')}]$

$$\sigma_i(\mathcal{A}^*y) \begin{cases} = 1, & \text{if } \sigma_i(W) \neq 0, \\ \in [0, 1), & \text{if } \sigma_i(W) = 0. \end{cases}$$

The function

$$f(X) = \|X\|_* - y^T[\mathcal{A}X - \mathcal{A}W] = \|X\|_* - \langle \mathcal{A}^*y, X \rangle + \|W\|_*$$

becomes the objective function of (4) on the feasible set of (4). Note that $\langle \mathcal{A}^*y, X \rangle \leq \|X\|_*$ by $\|\mathcal{A}^*y\| \leq 1$ and the definition of dual norm. So, $f(X) \geq \|X\|_* - \|X\|_* + \|W\|_* = \|W\|_*$ and this function attains its unconstrained minimum in X at $X = W$. Hence $X = W$ is an optimal solution to (4). It remains to show that this optimal solution is unique. Let Z be another optimal solution to the problem. Then $f(Z) - f(W) = \|Z\|_* - y^T\mathcal{A}Z = \|Z\|_* - \langle \mathcal{A}^*y, Z \rangle = 0$. This together with the fact $\|\mathcal{A}^*y\| \leq 1$ imply that there exist SVDs for \mathcal{A}^*y and Z such that:

$$\mathcal{A}^*y = \tilde{U} \text{Diag}(\sigma(\mathcal{A}^*y))\tilde{V}^T, \quad Z = \tilde{U} \text{Diag}(\sigma(Z))\tilde{V}^T,$$

where $\tilde{U} \in \mathbb{R}^{m \times r}$ and $\tilde{V} \in \mathbb{R}^{n \times r}$ are orthogonal matrices, and $\sigma_i(Z) = 0$ if $\sigma_i(\mathcal{A}^*y) \neq 1$. Thus, for $\sigma_i(\mathcal{A}^*y) = 0, \forall i \in \{s' + 1, \dots, r\}$, we must have $\sigma_i(Z) = \sigma_i(W) = 0$. By the two forms of SVDs of \mathcal{A}^*y as above, $U_{m \times s'}V_{n \times s'}^T = \tilde{U}_{m \times s'}\tilde{V}_{n \times s'}^T$ where $\tilde{U}_{m \times s'}, \tilde{V}_{n \times s'}^T$ are the corresponding submatrices of \tilde{U}, \tilde{V} , respectively. Without loss of generality, let

$$U = [u_1, u_2, \dots, u_r], \quad V = [v_1, v_2, \dots, v_r] \quad \text{and} \quad \tilde{U} = [\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_r], \quad \tilde{V} = [\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_r],$$

where $u_j = \tilde{u}_j$ and $v_j = \tilde{v}_j$ for the corresponding index $j \in \{i : \sigma_i(\mathcal{A}^*y) = 0, i \in \{s' + 1, \dots, r\}\}$. Then we have

$$Z = \sum_{i=1}^{s'} \sigma_i(Z) \tilde{u}_i \tilde{v}_i^T, \quad W = \sum_{i=1}^{s'} \sigma_i(W) u_i v_i^T.$$

From $U_{m \times s'}V_{n \times s'}^T = \tilde{U}_{m \times s'}\tilde{V}_{n \times s'}^T$, we obtain that

$$\sum_{i=s'+1}^r \sigma_i(\mathcal{A}^*y) \tilde{u}_i \tilde{v}_i^T = \sum_{i=s'+1}^r \sigma_i(\mathcal{A}^*y) u_i v_i^T.$$

Therefore, we deduce

$$\begin{aligned} & \sum_{i=s'+1, \sigma_i(\mathcal{A}^*y) \neq 0}^r \sigma_i(\mathcal{A}^*y) \tilde{u}_i \tilde{v}_i^T + \sum_{i=s'+1, \sigma_i(\mathcal{A}^*y) = 0}^r \tilde{u}_i \tilde{v}_i^T \\ &= \sum_{i=s'+1, \sigma_i(\mathcal{A}^*y) \neq 0}^r \sigma_i(\mathcal{A}^*y) u_i v_i^T + \sum_{i=s'+1, \sigma_i(\mathcal{A}^*y) = 0}^r u_i v_i^T \\ &=: \Omega. \end{aligned}$$

Clearly, the rank of Ω is no less than $r - s' \geq r - s$. From the orthogonality property of U, V and \tilde{U}, \tilde{V} , we easily derive that

$$\Omega^T \tilde{u}_i \tilde{v}_i^T = 0, \quad \Omega^T u_i v_i^T = 0, \quad \text{for all } i \in \{1, 2, \dots, s'\}.$$

Thus, we obtain $\Omega^T(Z - W) = 0$, which implies that the rank of the matrix $Z - W$ is no more than s . Since $\gamma_s(\mathcal{A}) < 1$, there exists \tilde{y} such that

$$\sigma_i(\mathcal{A}^*\tilde{y}) \begin{cases} = 1, & \text{if } \sigma_i(Z - W) \neq 0, \\ \in [0, 1), & \text{if } \sigma_i(Z - W) = 0. \end{cases}$$

Therefore, $0 = \tilde{y}^T \mathcal{A}(Z - W) = \langle \mathcal{A}^*\tilde{y}, Z - W \rangle = \|Z - W\|_*$. Then $Z = W$. \square

For the G -number $\hat{\gamma}_s(\mathcal{A})$, we directly obtain the following equivalent theorem of s -goodness from Proposition 2.6 and Theorem 3.1.

Theorem 3.2. *Let $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ be a linear transformation, and $s \in \{1, 2, \dots, r\}$. Then \mathcal{A} is s -good if and only if $\hat{\gamma}_s(\mathcal{A}) < 1/2$.*

For $X \in \mathbb{R}^{m \times n}$, we define the sum of the s largest singular values of X as

$$\|X\|_{s,*} := \max_{Z \in P_s} \langle Z, X \rangle.$$

We immediately obtain the following result utilizing Proposition 2.6 and Theorem 3.2.

Corollary 3.3. *Let $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ be a linear transformation, and $s \in \{1, 2, \dots, r\}$. Then $\hat{\gamma}_s(\mathcal{A})$ is the best upper bound on the norm $\|X\|_{s,*}$ of matrices $X \in \text{Null}(\mathcal{A})$ such that $\|X\|_* \leq 1$. As a result, the linear transformation \mathcal{A} is s -good if and only if the maximum of $\|\cdot\|_{s,*}$ -norms of matrices $X \in \text{Null}(\mathcal{A})$ with $\|X\|_* = 1$ is less than $1/2$.*

4. EXACT AND STABLE RECOVERY VIA G -NUMBER

In the previous sections, we showed that G -numbers $\gamma_s(\mathcal{A})$ and $\hat{\gamma}_s(\mathcal{A})$ are responsible for s -goodness of a linear transformation \mathcal{A} . Observe that the definition of s -goodness of a linear transformation \mathcal{A} indicates that whenever the observation b in the following

$$(19) \quad \hat{W} \in \operatorname{argmin}_X \{\|X\|_* : \|\mathcal{A}X - b\| \leq \varepsilon\}$$

is exact (noiseless) and comes from a s -rank matrix W such that $b = \mathcal{A}W$, W is the unique optimal solution of the above optimization problem (19) where ε is set to 0. This establishes a sufficient condition for the precise LMR of an s -rank matrix W in the “ideal case” when there is no measurement error or noise and the optimization problem (4) is solved exactly.

Theorem 4.1. *Let $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ be a linear transformation, and $s \in \{1, 2, \dots, r\}$. Let W be a s -rank matrix such that $\mathcal{A}W = b$. If \mathcal{A} is s -good ($\hat{\gamma}_s(\mathcal{A}) < 1/2$, or $\gamma_s(\mathcal{A}) < 1$), then W is the unique solution to LMR (1), i.e., the solution to LMR (1) can be exactly recovered from Problem (4).*

Proof. By the definition of s -goodness of a linear transformation \mathcal{A} , the assumptions that $\mathcal{A}W = b$ and $\operatorname{rank}(W) \leq s$ imply that W is the unique solution to problem (4). It remains to show that W is the unique solution to problem (1). For a contradiction, suppose there is another solution Y to problem (1). Then $\mathcal{A}W = \mathcal{A}Y = b$. By the s -goodness of \mathcal{A} , the problem $\min\{\|X\|_* : \mathcal{A}X = \mathcal{A}W\} \approx \min\{\|X\|_* : \mathcal{A}X = \mathcal{A}Y\}$ has a unique solution, hence $Y = W$ and we reached a contradiction. \square

It turns out that the same quantities $\gamma_s(\mathcal{A})$ ($\hat{\gamma}_s(\mathcal{A})$) can be used to measure the error of low-rank matrix recovery in the case when the matrix $W \in \mathbb{R}^{m \times n}$ is not s -rank and the problem (4) is not solved exactly. In what follows, let $W = U\operatorname{Diag}(\sigma(W))V^T$, where $\sigma(W) = (\sigma_1(W), \dots, \sigma_r(W))^T$ and $\sigma_1(W) \geq \dots \geq \sigma_r(W) \geq 0$ are the singular values of W in nonincreasing order. Let $W^s := U\operatorname{Diag}((\sigma_1(W), \dots, \sigma_s(W), 0, \dots, 0)^T)V^T$. Clearly, in terms of nuclear norm, W^s stands for the best s -rank approximation of W . In order to establish the error bound in the “non-ideal case”, we also need the following assumption for a matrix $X \in \mathbb{R}^{m \times n}$:

Block Assumption: We say that X satisfies the *block assumption* with respect to W if there exists $(U, V) \in \Xi(W)$ such that $U^T X V$ has the block form as

$$U^T X V = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix},$$

where $X_1 \in \mathbb{R}^{s \times s}$ and $X_2 \in \mathbb{R}^{(r-s) \times (r-s)}$. In this case, we write $X^{(s)} := U \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} V^T$.

Theorem 4.2. Let $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ be a linear transformation, $s \in \{0, 1, 2, \dots, r\}$, and $\hat{\gamma}_s(\mathcal{A}) < 1/2$ (or, equivalently, $\gamma_s(\mathcal{A}) < 1$). Also let W be a matrix such that $\mathcal{A}W = b$. Let X be a v -optimal solution to the problem (4), meaning that

$$\mathcal{A}X = \mathcal{A}W \text{ and } \|X\|_* \leq \text{Opt}(\mathcal{A}W) + v,$$

where $\text{Opt}(\mathcal{A}W)$ is the optimal value of (4). If the Block Assumption holds for X , then

$$\|X - W\|_* \leq \frac{v + 2\|W - W^s\|_*}{1 - 2\hat{\gamma}_s(\mathcal{A})} = \frac{1 + \gamma_s(\mathcal{A})}{1 - \gamma_s(\mathcal{A})} [v + 2\|W - W^s\|_*].$$

Proof. Set $Z := X - W$. Let $D_1 := \text{Diag}((\sigma_1(W), \dots, \sigma_s(W))^T)$, $D_2 := \text{Diag}((\sigma_{s+1}(W), \dots, \sigma_r(W))^T)$. Using the assumptions, we obtain that Z has the form

$$Z = U \begin{pmatrix} X_1 - D_1 & 0 \\ 0 & X_2 - D_2 \end{pmatrix} V^T.$$

Define

$$Z^{(s)} := U \begin{pmatrix} X_1 - D_1 & 0 \\ 0 & 0 \end{pmatrix} V^T.$$

It is easy to verify that $Z^{(s)} = X^{(s)} - W^s$ and $\|Z^{(s)}\|_* \leq \|Z\|_{s,*}$. Along with the fact $\mathcal{A}Z = 0$ and Corollary 3.3, this yields

$$(20) \quad \|Z^{(s)}\|_* \leq \|Z\|_{s,*} \leq \hat{\gamma}_s(\mathcal{A}) \|Z\|_*.$$

On the other hand, W is a feasible solution to (4), so $\text{Opt}(\mathcal{A}W) \leq \|W\|_*$. Thus, we have

$$(21) \quad \begin{aligned} \|W\|_* + v &\geq \|W + Z\|_* \geq \|W^s + Z - Z^{(s)}\|_* - \|Z^{(s)} + W - W^s\|_* \\ &= \|W^s\|_* + \|Z - Z^{(s)}\|_* - \|Z^{(s)}\|_* - \|W - W^s\|_*, \end{aligned}$$

where the last equation follows from the facts that $W^s(Z - Z^{(s)})^T = 0 = (W - W^s)(Z^{(s)})^T$ and $(W^s)^T(Z - Z^{(s)}) = 0 = (W - W^s)^T Z^{(s)}$, and Lemma 2.3 in [33]. This is equivalent to

$$\|Z - Z^{(s)}\|_* \leq \|Z^{(s)}\|_* + 2\|W - W^s\|_* + v.$$

Therefore, we obtain

$$\begin{aligned} \|Z\|_* &\leq \|Z^{(s)}\|_* + \|Z - Z^{(s)}\|_* \leq 2\|Z^{(s)}\|_* + 2\|W - W^s\|_* + v \\ &\leq 2\hat{\gamma}_s(\mathcal{A})\|Z\|_* + 2\|W - W^s\|_* + v. \end{aligned}$$

Since $\hat{\gamma}_s(\mathcal{A}) < 1/2$, we reach the desired conclusion. \square

Notice that the above Block Assumption holds naturally in the SSR (CMP) context. In general, we may have

$$U^T X V = \begin{pmatrix} X_1 & X_3 \\ X_4 & X_2 \end{pmatrix},$$

where either X_3 or X_4 is not zero. In this case, we have

$$Z = U \begin{pmatrix} X_1 - D_1 & X_3 \\ X_4 & X_2 - D_2 \end{pmatrix} V^T.$$

If we define

$$Z^{(s)} := U \begin{pmatrix} X_1 - D_1 & 0 \\ 0 & 0 \end{pmatrix} V^T,$$

we cannot conclude (21). If we define

$$Z^{(s)} := U \begin{pmatrix} X_1 - D_1 & X_3 \\ X_4 & 0 \end{pmatrix} V^T,$$

we cannot conclude $\|Z^{(s)}\|_* \leq \|Z\|_{s,*}$. It is not difficult to give counterexamples to illustrate the above facts. Meanwhile, in the last two cases, the rank of $Z^{(s)}$ may be greater than s . Thus the

condition $\hat{\gamma}_s(\mathcal{A}) < 1/2$ is not sufficient, and hence we need more strict restrictions on the linear transformation \mathcal{A} .

Below, we consider approximate solutions X to the problem

$$(22) \quad \text{Opt}(b) = \min_{X \in \mathbb{R}^{m \times n}} \{\|X\|_* : \|\mathcal{A}X - b\| \leq \varepsilon\}$$

where $\varepsilon \geq 0$ and $b = \mathcal{A}W + \zeta$, $\zeta \in \mathbb{R}^p$ with $\|\zeta\| \leq \varepsilon$. We will show that in the “non-ideal case”, when W is “nearly s -rank” and (22) is solved to near-optimality, the error of the LMR via NNM can be measured by $\hat{\gamma}_s(\mathcal{A}, \beta)$ with a finite β .

Theorem 4.3. *Let $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ be a linear transformation, and $s \in \{1, 2, \dots, r\}$, and let $\beta \in [0, +\infty]$ such that $\hat{\gamma} := \hat{\gamma}_s(\mathcal{A}, \beta) < 1/2$ (or $\gamma := \gamma_s(\mathcal{A}, \beta/(1 - \hat{\gamma})) < 1$). Let $\varepsilon \geq 0$ and let W and b in (22) be such that $\|\mathcal{A}W - b\| \leq \varepsilon$, and let W^s be defined in the beginning of this section. Let X be a (ϑ, v) -optimal solution to the problem (22), meaning that*

$$\|\mathcal{A}X - b\| \leq \vartheta \text{ and } \|X\|_* \leq \text{Opt}(b) + v.$$

If the Block Assumption holds for X , then

$$(23) \quad \begin{aligned} \|X - W\|_* &\leq \frac{2\beta(\vartheta + \varepsilon) + 2\|W - W^s\|_* + v}{1 - 2\hat{\gamma}} \\ &= \frac{1 + \gamma}{1 - \gamma} [2\beta(\vartheta + \varepsilon) + 2\|W - W^s\|_* + v]. \end{aligned}$$

Proof. Note that W is a feasible solution to (22). Let $Z = X - W$. As in the proof of Theorem 4.2, we obtain that $\|Z^{(s)}\|_* \leq \|Z\|_{s,*}$ and

$$\|Z\|_* \leq 2\|Z^{(s)}\|_* + 2\|W - W^s\|_* + v.$$

Employing (14) in Theorem 2.7, we derive

$$(24) \quad \|Z\|_{s,*} \leq \beta\|\mathcal{A}Z\| + \hat{\gamma}\|Z\|_* \leq \beta(\vartheta + \varepsilon) + \hat{\gamma}\|Z\|_*,$$

where the last inequality holds by $\|\mathcal{A}Z\| = \|\mathcal{A}X - b + b - \mathcal{A}Z\| \leq \|\mathcal{A}X - b\| + \|b - \mathcal{A}Z\|$. Combining with the above inequalities, we obtain

$$\|Z\|_* \leq 2\beta(\vartheta + \varepsilon) + 2\hat{\gamma}\|Z\|_* + 2\|W - W^s\|_* + v.$$

Now, the desired conclusion follows from the assumption $\hat{\gamma} < 1/2$ and $\gamma = \hat{\gamma}/(1 + \hat{\gamma})$. \square

Theorem 4.3 shows that under the Block Assumption the error bound (23) for imperfect low-rank matrix recovery can be bounded in terms of $\hat{\gamma}_s(\mathcal{A}, \beta)$, β , measurement error ε , “s-tail” $\|W - W^s\|_*$ and the accuracy (ϑ, v) to which the estimate solves the program (22). Note that we need $\gamma_s(\mathcal{A}, \beta) < 1$ (or $\hat{\gamma}_s(\mathcal{A}, \beta) < 1/2$). However, the “true” necessary and sufficient condition for s -goodness is $\gamma_s(\mathcal{A}) < 1$ (or $\hat{\gamma}_s(\mathcal{A}) < 1/2$). Also, note that $\gamma_s(\mathcal{A}, \beta) = \gamma_s(\mathcal{A})$ for all finite “large enough” values of β , see Proposition 2.3 for details.

5. COMPUTING BOUNDS ON THE G -NUMBER VIA CONVEX OPTIMIZATION

We showed that G -number $\hat{\gamma}_s(\mathcal{A}, \beta)$ controls some of the fundamental properties of a linear transformation \mathcal{A} relative to LMR. Since it seems difficult to evaluate these quantities exactly, we will provide ways of computing upper and lower bounds on these quantities $\hat{\gamma}_s(\mathcal{A}, \beta)$ via convex optimization techniques.

5.1. Computing lower bounds on $\hat{\gamma}_s(\mathcal{A}, \beta)$. Note that $\hat{\gamma}_s(\mathcal{A}, \beta) \geq \hat{\gamma}_s(\mathcal{A})$ for any $\beta > 0$ by Proposition 2.4. Therefore, we may establish a lower bound for \hat{G} -numbers $\hat{\gamma}_s(\mathcal{A}, \beta)$ by giving such a bound for $\hat{\gamma}_s(\mathcal{A})$. We can bound $\hat{\gamma}_s(\mathcal{A})$ from below utilizing Theorem 2.7. Recall von Neumann's trace inequality [30]: $\langle Y, Z \rangle \leq \langle \sigma(Y), \sigma(Z) \rangle$ for every pair of matrices $Y, Z \in \mathbb{R}^{m \times n}$, where the equality holds when Y, Z share the same orthogonal row and column spaces. In what follows, we define

$$\Xi(\mathcal{A}) := \{(U, V) : U \in \mathbb{R}^{m \times r}, V \in \mathbb{R}^{n \times r}, \exists W = U \text{Diag}(\sigma(W))V^T, \mathcal{A}W = 0\}.$$

From the representation (15), we obtain

$$\hat{\gamma}(\mathcal{A}) = \max_{\Sigma \in P_s} f(\Sigma), \quad f(\Sigma) = \max_X \{\langle \Sigma, X \rangle : \|X\|_* \leq 1, \mathcal{A}X = 0\}.$$

It is easy to see that $f(\Sigma)$ is convex. Then, we solve the convex optimization problem

$$(25) \quad X_\Sigma \in \operatorname{argmax}_X \{\langle \Sigma, X \rangle : \|X\|_* \leq 1, \mathcal{A}X = 0\},$$

we obtain a linear form $\langle X_\Sigma, \Theta \rangle$ of $\Theta \in P_s$ which under-estimates $f(\Theta)$ everywhere and agrees with $f(\Theta)$ when $\Theta = \Sigma$. Notice that

$$\begin{aligned} & \max_X \{\langle \Sigma, X \rangle : \|X\|_* \leq 1, \mathcal{A}X = 0\} \\ & \geq \max_{X, (U, V) \in \Xi(\mathcal{A})} \{\langle \Sigma, X \rangle : \|X\|_* \leq 1, \mathcal{A}X = 0, \Sigma = U \text{Diag}(t)V^T, X_\Sigma = U \text{Diag}(x_t)V^T\}. \end{aligned}$$

Since we need only to focus the lower bound via the above problem (25), in this sense, we may set $\Sigma = U \text{Diag}(t)V^T$ by choosing $(U, V) \in \Xi(\mathcal{A})$ and $t \in \mathbb{R}^r$ with $\|t\|_1 \leq s, \|t\|_\infty \leq 1$. Thus, we may obtain a lower bound from the following optimization problem:

$$\max_{x_t} \{\langle t, x_t \rangle : \|x_t\|_1 \leq 1, \mathcal{A}[U \text{Diag}(x_t)V^T] = 0\}.$$

For simplicity, we define \mathcal{A} by a set of p matrices $A_i \in \mathbb{R}^{m \times n}, i \in \{1, 2, \dots, p\}$:

$$\mathcal{A}(\cdot) = (\langle A_1, \cdot \rangle, \langle A_2, \cdot \rangle, \dots, \langle A_p, \cdot \rangle)^T.$$

Thus, we may rewrite

$$(26) \quad \mathcal{A}X_\Sigma = Ax_t$$

where $A \in \mathbb{R}^{p \times r}$ with $A_{ij} = (U^T A_i V)_{jj}$. In this sense, we may formulate the convex optimization problem (25) as the following group of LP problems

$$(27) \quad x_t \in \operatorname{argmax}_x \{\langle t, x \rangle : \|x\|_1 \leq 1, Ax = 0\}.$$

The optimal solutions may not be unique because for a given Σ orthogonal matrices $U \in \mathbb{R}^{m \times r}, V \in \mathbb{R}^{n \times r}$ are usually not unique. In order to establish a lower bound for $\hat{\gamma}_s(\mathcal{A})$, we may choose one pair $(U, V) \in \Xi(\mathcal{A})$ and then solve the corresponding LP (27). We obtain a linear form $v^T x_t$ of $v \in \Delta_s$ where

$$\Delta_s := \{x \in \mathbb{R}^r : \|x\|_1 \leq s, \|x\|_\infty \leq 1\}.$$

Therefore, we obtain a lower bound result on $\hat{\gamma}_s(\mathcal{A})$ as follows:

Proposition 5.1. *Let \mathcal{A} be specified as above and x_t given by (27). Then, $\max_{v \in \Delta_s} v^T x_t$ is a lower bound on $\hat{\gamma}_s(\mathcal{A})$.*

Clearly, the above bound is easily computable. As in [18], we can use the standard sequential convex approximation scheme for maximizing the convex function $f(\cdot)$ over P_s . In particular, we can run the iterative process

$$t_{k+1} \in \operatorname{argmax}_{v \in \Delta_s} v^T x_{t_k}, \quad t_1 \in \Delta_s, \quad U \text{Diag}(t_1)V^T \in P_s.$$

This leads to a monotone nondecreasing sequence of lower bounds $t_k^T x_{t_k}$ on $\hat{\gamma}_s(\mathcal{A})$. We may choose to terminate this iterative process when the improvement in the bounds falls below a given tolerance, and we can start several runs from randomly chosen points t_1 and orthogonal matrices $(U, V) \in \Xi(\mathcal{A})$.

5.2. Computing upper bounds on $\hat{\gamma}_s(\mathcal{A}, \beta)$. For an arbitrary linear transformation \mathcal{B} , we have

$$(28) \quad \begin{aligned} & \max_{\Sigma, X} \{ \langle \Sigma, X \rangle : \|X\|_* \leq 1, \mathcal{A}X = 0, \Sigma \in P_s \} \\ &= \max_{\Sigma, X} \{ \langle \Sigma, X - \mathcal{B}^* \mathcal{A}X \rangle : \|X\|_* \leq 1, \mathcal{A}X = 0, \Sigma \in P_s \}. \end{aligned}$$

In the same way as in (26), we define \mathcal{B} by a set of p matrices $B_k \in \mathbb{R}^{m \times n}$, $k \in \{1, 2, \dots, p\}$ and \mathcal{B}^* as

$$\mathcal{B}^*(u) = \sum_{k=1}^p u_k B_k, \quad u = (u_1, u_2, \dots, u_p)^T \in \mathbb{R}^p.$$

For simplicity, suppose (26) holds. Using a similar analysis, we choose all B_j (simultaneously diagonalizable) such that they have the singular value decompositions $B_k = U \text{Diag}(y_k) V^T$ ($y_k \in \mathbb{R}^r$) and then rewrite (28) as

$$(29) \quad \begin{aligned} & \max_{\Sigma, X} \{ \langle \Sigma, X - \mathcal{B}^* \mathcal{A}X \rangle : \|X\|_* \leq 1, \mathcal{A}X = 0, \Sigma \in P_s \} \\ &= \max_{t, x, U, V} \{ \langle t, x - B^T A x \rangle : \|x\|_1 \leq 1, A x = 0, t \in \Delta_s \}, \end{aligned}$$

where $B^T := [y_1, y_2, \dots, y_p]$. If we fix U, V , the above problem is easy to solve as it was done in [18]. In this case,

$$(30) \quad \begin{aligned} & \max_{t, x} \{ \langle t, x - B^T A x \rangle : \|x\|_1 \leq 1, A x = 0, t \in \Delta_s \} \\ & \leq \max_{t, x} \{ \langle t, x - B^T A x \rangle : \|x\|_1 \leq 1, t \in \Delta_s \} \\ &= \max_{t, i \in \{1, \dots, r\}} \{ \langle t, (I - B^T A) e_i \rangle : t \in \Delta_s \} \\ &= \max_{i \in \{1, \dots, r\}} \max_{t \in \Delta_s} \{ \langle t, (I - B^T A) e_i \rangle \} = \max_{i \in \{1, \dots, r\}} \|(I - B^T A) e_i\|_{s,1}, \end{aligned}$$

where $\|x\|_{s,1}$ is the sum of the s largest magnitudes of entries in x . Therefore, we have for all $B \in \mathbb{R}^{p \times r}$

$$\begin{aligned} \hat{\gamma}_s(\mathcal{A}) &= \max_{\Sigma, X} \{ \langle \Sigma, X \rangle : \|X\|_* \leq 1, \mathcal{A}X = 0, \Sigma \in P_s \} \\ &\leq \max_{U, V, i \in \{1, \dots, r\}} \|(I - B^T A) e_i\|_{s,1} =: f_{\mathcal{A},s}(B). \end{aligned}$$

Taking $\Gamma_s(\mathcal{A}, +\infty) := \min_B f_{\mathcal{A},s}(B)$, we obtain

$$\hat{\gamma}_s(\mathcal{A}) \leq \Gamma_s(\mathcal{A}, +\infty).$$

Observe that $f_{\mathcal{A},s}(B)$ is an easy-to-compute convex function of B for fixed U, V and it is indeed related to a semi-infinite programming [3]. Therefore, one may choose to utilize computational semi-infinite programming techniques to compute the quantity $\Gamma_s(\mathcal{A}, +\infty)$.

The above analysis motivates the following useful function of \mathcal{A} and β .

Definition 5.2. Let \mathcal{A} and the corresponding matrices $A_i, i \in \{1, 2, \dots, p\}$ be given as above. Let $\beta \in [0, +\infty]$. We define $\Gamma_s(\mathcal{A}, \beta)$ as follows:

$$(31) \quad \Gamma_s(\mathcal{A}, \beta) := \min_B \left\{ \max_{U, V, i \in \{1, \dots, r\}} \|(I - B^T A) e_i\|_{s,1} : \|(B)_{\cdot j}\|_d \leq \beta, 1 \leq j \leq r \right\},$$

where A is the matrix defined by A_i and U, V (as above), $(B)_{\cdot j}$ is the j th column of B . If there does not exist such a matrix B as above, we take $\Gamma_s(\mathcal{A}, \beta) = +\infty$. For convenience, we abbreviate the notation $\Gamma_s(\mathcal{A}, +\infty)$ to $\Gamma_s(\mathcal{A})$.

By modifying the above process, we obtain that $\Gamma_s(\mathcal{A}, \beta)$ provides an upper bound for \hat{G} -numbers $\hat{\gamma}_s(\mathcal{A}, \beta)$. Moreover, $\Gamma_s(\mathcal{A}, \beta)$ shares some properties similar to those of \hat{G} -numbers $\hat{\gamma}_s(\mathcal{A}, \beta)$. In other words, $\Gamma_s(\mathcal{A}, \beta)$ is nondecreasing in s , convex and nonincreasing in β , and is such that $\Gamma_s(\mathcal{A}, \beta) = \Gamma_s(\mathcal{A})$ for all large enough values of β . The following result shows that $\Gamma_s(\mathcal{A}, \beta)$ is an upper bound on $\hat{\gamma}_s(\mathcal{A}, \beta)$.

Theorem 5.3. *For every \mathcal{A} and $\beta \in [0, +\infty]$, we have $\Gamma_s(\mathcal{A}, \beta) \geq \hat{\gamma}_s(\mathcal{A}, \beta)$.*

Proof. Let W be a s -rank matrix with all nonzero singular values equal to 1 such that $W = U \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} V^T$, where I_s is the $s \times s$ identity matrix. For U, V , we get $\mathcal{A}W = A\sigma(W)$ where A is specified as in (26). Let $Y = [y_1, y_2, \dots, y_r] \in \mathbb{R}^{p \times r}$ be such that $\|y_i\|_d \leq \beta$ and the columns in $I - Y^T A$ are of the $\|\cdot\|_{s,1}$ -norm not exceeding $\Gamma_s(\mathcal{A}, \beta)$. Define the linear transformation \mathcal{B} such that $\mathcal{B}W := Y\sigma(W)$. Setting $y = Y\sigma(W)$, the fact that $\|y_i\|_d \leq \beta$, $i \in \{1, 2, \dots, r\}$ implies that $\|y\|_* \leq \beta\|\sigma(W)\|_1 \leq \beta s$. Furthermore, noting that $\sigma(W)$ is a s -sparse vector, we obtain

$$\|W - \mathcal{A}^* y\| = \|W - \mathcal{A}^* \mathcal{B}W\| = \|(I - B^T A)^T \sigma(W)\| \leq \Gamma_s(\mathcal{A}, \beta).$$

The desired conclusion follows immediately. \square

Note that $\|X\|_{st,*} \leq s\|X\|_{t,*}$ for all positive integers s, t . Thus, we may replace $\Gamma_s(\mathcal{A}, \beta)$ as $s\Gamma_1(\mathcal{A}, \beta)$, i.e.,

$$\hat{\gamma}_s(\mathcal{A}, \beta) \leq \Gamma_s(\mathcal{A}, \beta) \leq s\Gamma_1(\mathcal{A}, \beta).$$

Moreover, we have $\Gamma_1(\mathcal{A}, \beta) = \max_i \Upsilon_i$, where

$$(32) \quad \Upsilon_i := \min_{U, V, y_i} \{ \|e_i - A^T y_i\|_\infty : \|y_i\|_d \leq \beta \}, \quad i \in \{1, 2, \dots, r\}.$$

By direct calculation, note that the matrix A is the representation of \mathcal{A} with respect to U, V , we obtain

$$\begin{aligned} \Upsilon_i &= \min_{U, V, y} \max_j \{ |(e_i - A^T y)_j| : \|y\|_d \leq \beta \} \\ &= \min_{U, V, y} \max_x \{ \langle e_i - A^T y, x \rangle : \|y\|_d \leq \beta, \|x\|_1 \leq 1 \} \\ &= \max_X \min_y \{ \langle U e_i V^T, X \rangle - \langle \mathcal{A}^* y, X \rangle : \|y\|_d \leq \beta, \|X\|_* \leq 1, X = U \text{Diag}(x) V^T \} \\ &= \max_X \{ \langle U e_i V^T, X \rangle - \beta \|\mathcal{A} X\| : \|X\|_* \leq 1 \}. \end{aligned}$$

It follows from Theorem 2.7 that $\Upsilon_i \leq \hat{\gamma}_1(\mathcal{A}, \beta)$. Therefore, by Theorem 5.3, we have that the relaxation for $\hat{\gamma}_1(\mathcal{A}, \beta)$ is exact, i.e.,

$$(33) \quad \Gamma_1(\mathcal{A}, \beta) = \hat{\gamma}_1(\mathcal{A}, \beta).$$

As in Proposition 2.3, we present the following simple result which shows how large β needs to be to guarantee $\Gamma_s(\mathcal{A}, \beta) = \Gamma_s(\mathcal{A})$.

Proposition 5.4. *Let $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ be a linear transformation, $\beta \in [0, +\infty]$ and $s \in \{0, 1, 2, \dots, r\}$. For some $\rho > 0$, let the image of the unit $\|\cdot\|_*$ -ball in $\mathbb{R}^{m \times n}$ under the mapping $X \mapsto \mathcal{A}X$ contain the ball $B = \{x \in \mathbb{R}^p : \|x\|_1 \leq \rho\}$. Then for every $s \leq r$*

$$\beta \geq \frac{3}{2\rho} \text{ and } \Gamma_s(\mathcal{A}) < \frac{1}{2} \Rightarrow \Gamma_s(\mathcal{A}, \beta) = \Gamma_s(\mathcal{A}).$$

Proof. Fix $s \in \{1, 2, \dots, r\}$. Let $\Gamma_s(\mathcal{A}) < 1/2$. Then $\gamma := \hat{\gamma}_s(\mathcal{A}) < 1/2$ and hence for every matrix $W \in \mathbb{R}^{m \times n}$ with s nonzero singular values, equal to 1, there exists a vector $y \in \mathbb{R}^p$ such that

$$\|y\|_d \leq \beta \text{ and } \|\mathcal{A}^* y - X\| \leq \gamma.$$

By the triangle inequality, $\|\mathcal{A}^*y\| \leq 1 + \gamma < 3/2$. Following the same steps as in the proof of Proposition 2.3, we reach the desired conclusion. \square

6. S -GOODNESS AND RIP

We consider the connection between restricted isometry property and s -goodness of the linear transformation in LMR and present some explicit forms of restricted isometry (RI) constants and s -goodness constants, G -numbers. Recall that the s -restricted isometry constant δ_s of a linear transformation \mathcal{A} is defined as the smallest constant such that the following holds for all s -rank matrices $X \in \mathbb{R}^{m \times n}$

$$(34) \quad (1 - \delta_s)\|X\|_F^2 \leq \|\mathcal{A}X\|_2^2 \leq (1 + \delta_s)\|X\|_F^2.$$

In this case, we say \mathcal{A} possesses the $RI(\delta_s)$ -property (RIP) as in the CS context. For details, see [33, 10, 21, 27, 29] and the references therein.

6.1. $\hat{\gamma}_s(\mathcal{A})$ and δ_{2s} . We will show that the $RI(\delta_{2s})$ -property of \mathcal{A} implies that G -numbers satisfy $\hat{\gamma}_s(\mathcal{A}) < 1/2$ and $\gamma_s(\mathcal{A}) < 1$, which means that the RIP implies the sufficient conditions for s -goodness.

Theorem 6.1. *Let $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ be a linear transformation, and $s \in \{1, 2, \dots, r\}$. Assume that \mathcal{A} has RIP with $\delta_{2s} < \sqrt{2} - 1$, and let $\|\cdot\|_d := \|\cdot\|_2$ for vectors in \mathbb{R}^p . Then we have*

$$(35) \quad \hat{\gamma}_s(\mathcal{A}, \beta) \leq \frac{\sqrt{2}\delta_{2s}}{1 + (\sqrt{2} - 1)\delta_{2s}} < \frac{1}{2} \text{ for all } \beta \geq \frac{\sqrt{s(1 + \delta_{2s})}}{1 + (\sqrt{2} - 1)\delta_{2s}}.$$

This implies

$$(36) \quad \hat{\gamma}_s(\mathcal{A}) \leq \frac{\sqrt{2}\delta_{2s}}{1 + (\sqrt{2} - 1)\delta_{2s}} < \frac{1}{2} \text{ and } \gamma_s(\mathcal{A}) \leq \frac{\sqrt{2}\delta_{2s}}{1 - \delta_{2s}} < 1,$$

and hence \mathcal{A} is s -good.

Proof. By Theorem 2.7, in order to show (35), it is enough to verify that for all $X \in \mathbb{R}^{m \times n}$

$$(37) \quad \|X\|_{s,*} \leq \frac{\sqrt{s(1 + \delta_{2s})}}{1 + (\sqrt{2} - 1)\delta_{2s}} \|\mathcal{A}X\|_2 + \frac{\sqrt{2}\delta_{2s}}{1 + (\sqrt{2} - 1)\delta_{2s}} \|X\|_*.$$

Without loss of generality, let SVD of X be specified by

$$X = U \text{Diag}(x) V^T,$$

where $U \in \mathbb{R}^{m \times r}$ and $V \in \mathbb{R}^{n \times r}$, and $\sigma(X) := x = (x_1, \dots, x_r)^T$ is the vector of the singular values of X with $x_1 \geq \dots \geq x_r \geq 0$. We decompose x into a sum of vectors $x_{T_i}, i \in \{0, 1, 2, \dots\}$, each of sparsity at most s , where T_0 corresponds to the locations of the s largest entries of X , and T_1 to the locations of the next s largest entries, and so on (with except for the last part). We define $X_{T_i} := U \text{Diag}(x_{T_i}) V^T$. Then, X_{T_0} is the part of X corresponding to the s largest singular values, X_{T_1} is the part corresponding to the next s largest singular values, and so on. Clearly, $X_{T_0}, X_{T_1}, \dots, X_{T_i}, \dots$ are all orthogonal to one another, and $\text{rank}(X_{T_i}) \leq s$. From the above partition, we easily obtain that for $j \geq 2$,

$$\|X_{T_j}\|_F \leq s^{1/2} \|X_{T_j}\| \leq s^{-1/2} \|X_{T_{j-1}}\|_*.$$

Then it follows that

$$\sum_{j \geq 2} \|X_{T_j}\|_F \leq s^{-1/2} \sum_{j \geq 2} \|X_{T_{j-1}}\|_* \leq s^{-1/2} (\|X\|_* - \|X_{T_0}\|_*).$$

This yields

$$(38) \quad \|X - X_{T_0} - X_{T_1}\|_F = \left\| \sum_{j \geq 2} X_{T_j} \right\|_F \leq \sum_{j \geq 2} \|X_{T_j}\|_F \leq s^{-1/2} (\|X\|_* - \|X_{T_0}\|_*).$$

Noting that $\mathcal{A}(X_{T_0} + X_{T_1}) = \mathcal{A}(X - \sum_{j \geq 2} X_{T_j})$, we obtain

$$\begin{aligned} \|\mathcal{A}(X_{T_0} + X_{T_1})\|_2^2 &= \langle \mathcal{A}(X_{T_0} + X_{T_1}), \mathcal{A}(X - \sum_{j \geq 2} X_{T_j}) \rangle \\ &= \langle \mathcal{A}(X_{T_0} + X_{T_1}), \mathcal{A}X \rangle - \sum_{j \geq 2} \langle \mathcal{A}(X_{T_0} + X_{T_1}), \mathcal{A}X_{T_j} \rangle. \end{aligned}$$

From the RIP assumption of \mathcal{A} , we obtain that

$$\begin{aligned} |\langle \mathcal{A}(X_{T_0} + X_{T_1}), \mathcal{A}X \rangle| &\leq \|\mathcal{A}(X_{T_0} + X_{T_1})\|_2 \|\mathcal{A}X\|_2 \\ &\leq \sqrt{1 + \delta_{2s}} \|X_{T_0} + X_{T_1}\|_F \|\mathcal{A}X\|_2. \end{aligned}$$

By direct calculation,

$$\begin{aligned} \sum_{j \geq 2} |\langle \mathcal{A}(X_{T_0} + X_{T_1}), \mathcal{A}X_{T_j} \rangle| &\leq \sum_{j \geq 2} \delta_{2s} (\|X_{T_0}\|_F + \|X_{T_1}\|_F) \|X_{T_j}\|_F \\ &\leq \sqrt{2} \delta_{2s} \|X_{T_0} + X_{T_1}\|_F \sum_{j \geq 2} \|X_{T_j}\|_F, \end{aligned}$$

where the first inequality follows from Lemma 3.3 [10], and the second one follows from the inequality $(\|X_{T_0}\|_F + \|X_{T_1}\|_F)^2 \leq 2\|X_{T_0} + X_{T_1}\|_F^2$. Clearly, combining the RIP assumption on \mathcal{A} with the above inequalities, we have

$$\begin{aligned} (1 - \delta_{2s}) \|X_{T_0} + X_{T_1}\|_F^2 &\leq \langle \mathcal{A}(X_{T_0} + X_{T_1}), \mathcal{A}(X_{T_0} + X_{T_1}) \rangle \\ &\leq \sqrt{1 + \delta_{2s}} \|X_{T_0} + X_{T_1}\|_F \|\mathcal{A}X\|_2 + \sqrt{2} \delta_{2s} \|X_{T_0} + X_{T_1}\|_F \sum_{j \geq 2} \|X_{T_j}\|_F. \end{aligned}$$

This implies

$$(1 - \delta_{2s}) \|X_{T_0} + X_{T_1}\|_F \leq \sqrt{1 + \delta_{2s}} \|\mathcal{A}X\|_2 + \sqrt{2} \delta_{2s} \sum_{j \geq 2} \|X_{T_j}\|_F.$$

By (38) and the fact $\|X_{T_0}\|_* \leq \sqrt{s} \|X_{T_0}\|_F \leq \sqrt{s} \|X_{T_0} + X_{T_1}\|_F$, it follows that

$$\|X_{T_0}\|_* \leq \frac{\sqrt{s(1 + \delta_{2s})}}{1 - \delta_{2s}} \|\mathcal{A}X\|_2 + \frac{\sqrt{2} \delta_{2s}}{1 - \delta_{2s}} (\|X\|_* - \|X_{T_0}\|_*).$$

Noting that $\|X_{T_0}\|_* = \|X\|_{s,*}$, we establish (37), and hence we obtain the desired conclusion. \square

6.2. $\Gamma_s(\mathcal{A})$ and δ_{2s} . We consider the performance of $\Gamma_s(\mathcal{A})$ for s -goodness when \mathcal{A} has RIP. It turns out that this is similar to the CS case.

Theorem 6.2. *Let $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ be a linear transformation, and $s \in \{1, 2, \dots, r\}$. Assume that \mathcal{A} has RIP with $\delta_{ts} < 1$ for some positive constant t . Then we have*

$$(39) \quad \Gamma_1(\mathcal{A}) \leq \frac{\sqrt{2} \delta_{ts}}{(1 - \delta_{ts}) \sqrt{ts - 1}}.$$

Furthermore, if $s < \frac{(1 - \delta_{ts}) \sqrt{ts - 1}}{2\sqrt{2} \delta_{ts}}$, then $\Gamma_s(\mathcal{A}) \leq s \Gamma_1(\mathcal{A}) < 1/2$.

Proof. From Theorem 5.3, in order to establish the desired theorem, we only need to prove (39). By Theorem 2.7 and (33), it is enough to show that for every $X \in \mathbb{R}^{m \times n}$ with $\mathcal{A}X = 0$, we have

$$(40) \quad \|X\| = \|X\|_{1,*} \leq \hat{\gamma} \|X\|_*, \quad \hat{\gamma} := \hat{\gamma}_1(\mathcal{A}) \leq \frac{\sqrt{2} \delta_{ts}}{(1 - \delta_{ts}) \sqrt{ts - 1}}.$$

As in the proof of Theorem 6.1, let SVD of X be specified by

$$X = U \text{Diag}(x) V^T,$$

where $U \in \mathbb{R}^{m \times r}$ and $V \in \mathbb{R}^{n \times r}$, and $\sigma(X) := x = (x_1, \dots, x_r)^T$ is the vector of the singular values of X with $x_1 \geq \dots \geq x_r \geq 0$. Set $l = \lfloor ts/2 \rfloor$. We decompose x into a sum of vectors $x_{T_i}, i \in \{0, 1, 2, \dots\}$, where T_0 corresponds to the locations of the largest entries of X , T_1 to the locations of the next $l - 1$ largest entries, and $T_j (j \geq 2)$ to the locations of the next l largest entries, and so on, with evident modification for the last vector. We define $X_{T_i} := U \text{Diag}(x_{T_i}) V^T$. Then, X_{T_0} is the part of X corresponding to the largest singular values, X_{T_1} is the part corresponding to the next $l - 1$ largest singular values, and $X_{T_j} (j \geq 2)$ is the part corresponding to the next l largest singular values, and so on. From the above partition, we easily obtain that for $j \geq 2$,

$$\|X_{T_j}\|_F \leq l^{1/2} \|X_{T_j}\| \leq l^{-1/2} \|X_{T_{j-1}}\|_*.$$

Then it follows that

$$\sum_{j \geq 2} \|X_{T_j}\|_F \leq l^{-1/2} \sum_{j \geq 2} \|X_{T_{j-1}}\|_* \leq l^{-1/2} (\|X\|_* - \|X_{T_0}\|_*).$$

This yields

$$\|X - X_{T_0} - X_{T_1}\|_F = \left\| \sum_{j \geq 2} X_{T_j} \right\|_F \leq \sum_{j \geq 2} \|X_{T_j}\|_F \leq l^{-1/2} (\|X\|_* - \|X_{T_0}\|_*) \leq l^{-1/2} \|X\|_*.$$

Together with $\mathcal{A}X = 0$ and Lemma 3.3 [10], we obtain

$$\begin{aligned} 0 &= \langle \mathcal{A}(X_{T_0} + X_{T_1}), \mathcal{A}X \rangle \\ &= \langle \mathcal{A}(X_{T_0} + X_{T_1}), \mathcal{A}(X_{T_0} + X_{T_1}) \rangle + \langle \mathcal{A}(X_{T_0} + X_{T_1}), \mathcal{A}(X - X_{T_0} - X_{T_1}) \rangle \\ &\geq (1 - \delta_l) \|X_{T_0} + X_{T_1}\|_F^2 - l^{-1/2} \delta_{2l} \|X\|_*^2. \end{aligned}$$

This implies

$$(1 - \delta_l) \|X_{T_0} + X_{T_1}\|_F \leq l^{-\frac{1}{2}} \delta_{ts} \|X\|_*.$$

Note the facts that $\|X\|_{1,*} = \|X_{T_0}\|_* \leq \|X_{T_0}\|_F \leq \|X_{T_0} + X_{T_1}\|_F$ and $\delta_l \leq \delta_{2l} \leq \delta_{ts}$ because of $l \leq ts/2$. We then have

$$(1 - \delta_l) \|X_{T_0} + X_{T_1}\|_F \leq (1 - \delta_{ts}) \|X_{T_0} + X_{T_1}\|_F \leq \sqrt{\frac{2}{\delta_{ts}}} \delta_{ts} \|X\|_* \leq \sqrt{\frac{2}{\delta_{ts-1}}} \delta_{ts} \|X\|_*.$$

This proves (40) and hence the desired conclusion holds. \square

6.3. A bound for RIP. From Theorems 3.2 and 6.1, we actually provide a sufficient condition for s -goodness in terms of RI constant δ_{2s} : \mathcal{A} is s -good if it has the RIP with $\delta_{2s} < \sqrt{2} - 1$. This establishes a bound on the RI constant of \mathcal{A} .

Theorem 6.3. *Let $b = \mathcal{A}W$ for some given s -rank matrix W . If $\delta_{2s} < \sqrt{2} - 1$, then $W = X^*$ where X^* is the unique optimal solution to NNM.*

Recht et al. [33] showed that if $\delta_{5s} < 1/10$, then $X^* = W$ where X^* is the unique optimal solution to NNM. Lee and Bresler [21] gave $\delta_{3s} < 1/(1 + 4/\sqrt{3})$ by employing an analogue of the approach for SSR [9]; Candès and Plan [10] gave $\delta_{4s} < \sqrt{2} - 1$ based on the work [9, 13]; Mohan and Fazel [29] gave $\delta_{2s} < 0.307$, $\delta_{3s} < 2\sqrt{5} - 4$, and $\delta_{4s} < (8 - \sqrt{40})/3$ by combining a s, s' -restricted orthogonality constant property which extended the recent work in CS [5, 6, 7]. Meka, Jain and Dhillon [27] gave $\delta_{2s} < 1/3$ via singular value projection (SVP), though the efficient SVP algorithm requires a priori knowledge of the rank of W . Oymak, Mohan, Fazel and Hassibi [31] proposed a general technique for translating results from SSR to LMR, where they give the current best bound on the restricted isometry constant $\delta_{2s} < 0.472$. Our results were independently obtained.

7. CONCLUSION

In this paper, we studied the s -goodness characterization of the linear transformation in LMR. By employing the properties of G -numbers γ_s and $\hat{\gamma}_s$, we established necessary and sufficient conditions for a linear transformation to be s -good, and provided sufficient conditions for exact and stable LMR via NNM under mild assumptions. Furthermore, we obtained computable upper bounds of G -number $\hat{\gamma}_s$, which lead to verifiable sufficient conditions for exact LMR.

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